Analytic Theory on Probability*

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Abstract

We translate the beginning of the central chapter of the original treatise [1814Lapl], chapter 3 in the 2nd book, with its over 500 pages, and add some comments and annotations, in order that the interested reader of the 21th century may understand what is meant. In adjustment to the English style of today we replace the future tense by the present, analogously to the expectation for a back-translation from a Hebrew text. This translation is kept closely in style to the original text, so that the reader, who has got no knowledge of the French language of the 19th century, will get an impression of Laplace's style. In order to facilitate quotations, the page numbers of the French second edition of 1814 are placed within the translation at the places, where they are in the original, after the following full stop. The comments in the footnotes serve the subsequent explanation of the mathematical contents.

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3 The Laws of Probability, which Follow the Infinite Repetition of Events.

16. The corresponding probability of events develops more and more in the same way as the events are multiplied: The average results, the gains and the losses of these multiplications reach out for limits, to which they always approach with increasing probability. The determination of that approach and of its limits is one of the most interesting and delicate parts of chance analysis.

At first we consider the way the probability of two simple events develops, of which the one or the other necessarily results of each throw, when multiplicated many times. Obviously, the event with the greater possibility must take place more frequently in a preset number of throws; and of course each of the two events should take place proportionally

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to its possibility, if the throws are repeated many, many times. That could be supported by experience. That is an important theorem, which now we are going to prove analytically.

We have seen at No. 6, that if p and $1-p$ are the corresponding probabilities of both events a and b, the probability of $x+x'$ throws, that the event a will take place x times, and the event b x' times, is equal to:

$$\frac{1 \cdot 2 \cdot 3 \cdots (x+x')}{1 \cdot 2 \cdot 3 \cdots x \cdot 1 \cdot 2 \cdot 3 \cdots x'} \cdot p^x \cdot (1-p)^{x'} \; ; \tag{1}$$

this is the $(x'+1)^{\text{th}}$ summand of the binomial $[p+(1-p)]^{x+x'}$. Now let's consider the greatest of these summands, which we call k. The preceding summand be $\frac{k \cdot p}{1-p} \cdot \frac{x'}{x+1}$, and the subsequent summand be $k \cdot \frac{1-p}{p} \cdot \frac{x}{x'+1}$.

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For k being the greatest summand, there is a compelling demand, to fix simultaneously[1]:

$$\frac{p}{1-p} < \frac{x+1}{x'} > \frac{x}{x'+1} \; ; \tag{2}$$

from which easily can be concluded, that with $x+x'=n$ we receive[2]:

$$x < (n+1) \cdot p > (n+1) \cdot p - 1 \; ; \tag{3}$$

thus x is the greatest integer number, which is contained in $(n+1) \cdot p$; therefore results[3]:

$$x = (n+1) \cdot p - s \, , \tag{4}$$

of which follows:

$$p = \frac{x+s}{n+1} \, , \qquad 1-p = \frac{x'+1-s}{n+1} \, , \qquad \frac{p}{1-p} = \frac{x+s}{x'+1-s} \; ; \tag{5}$$

s shall be less than unity. If x and x' are huge numbers, then we receive in extemely good approximation:

$$\frac{p}{1-p} = \frac{x}{x'} \; ; \tag{6}$$

which means, that the exponents of p and of $1-p$ in the greater expression of the binomial are approached very much to each other in the ratio of the frequencies; thus the most probable combination (which can take place at a huge number n of throws) of all is the reason for each event occuring proportionally to its probability.

The l^{th} expression after the greatest is:

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots (x-l) \cdot 1 \cdot 2 \cdot 3 \cdots (x'+l)} \cdot p^{x-l} \cdot (1-p)^{x'+l} \; . \tag{7}$$

[1] today's syntax: $\frac{x}{x'+1} < \frac{p}{1-p} < \frac{x+1}{x'}$

[2] today's syntax: $(n+1) \cdot p - 1 < x < (n+1) \cdot p$

[3] Here, the expectation value $x = n \cdot \frac{x}{x+x'} = n \cdot p$ is rounded to be an integer.

Due to No. 32 of the first book we have[4]:

$$1 \cdot 2 \cdot 3 \cdots n \;=\; n^{n+\frac{1}{2}} \cdot c^{-n} \cdot \sqrt{2\,\pi} \cdot \left\{1 + \frac{1}{12 \cdot n} + \text{etc.}\right\}\;; \tag{8}$$

of which follows[5]:

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$$\frac{1}{1 \cdot 2 \cdot 3 \cdots (x-l)} \;=\; (x-l)^{l-x-\frac{1}{2}} \cdot \frac{c^{x-l}}{\sqrt{2\,\pi}} \cdot \left\{1 - \frac{1}{12 \cdot (x-l)} - \text{etc.}\right\}, \tag{9}$$

$$\frac{1}{1 \cdot 2 \cdot 3 \cdots (x'+l)} \;=\; (x'+l)^{-x'-l-\frac{1}{2}} \cdot \frac{c^{x'+l}}{\sqrt{2\,\pi}} \cdot \left\{1 - \frac{1}{12 \cdot (x'+l)} - \text{etc.}\right\}. \tag{10}$$

Now we develop the term $(x-l)^{l-x-\frac{1}{2}}$. Its hyperbolic logarithm is:

$$\left(l - x - \frac{1}{2}\right) \cdot \left[\log x + \log\left(1 - \frac{l}{x}\right)\right]\;; \tag{11}$$

but now is valid:

$$\log\left(1 - \frac{l}{x}\right) \;=\; -\frac{l}{x} - \frac{l^2}{2\,x^2} - \frac{l^3}{3\,x^3} - \frac{l^4}{4\,x^4} - \text{etc.}\,; \tag{12}$$

we neglect the set of the order $\frac{1}{n}$, and we assume, that l^2 does not exceed the order n at all; by this we neglect the terms of the order $\frac{l^4}{x^3}$, because x and x' belong to the order n. Therefore we receive:

$$\begin{aligned} &\left(l - x - \frac{1}{2}\right) \cdot \left[\log x + \log\left(1 - \frac{l}{x}\right)\right] \\ =\; &\left(l - x - \frac{1}{2}\right) \cdot \log x \;+\; l \;+\; \frac{l}{2\,x} \;-\; \frac{l^2}{2\,x} \;-\; \frac{l^3}{6\,x^2}\;; \end{aligned} \tag{13}$$

generating the following formula by inserting the logarithms to the numbers[6]:

$$(x-l)^{l-x-\frac{1}{2}} \;=\; c^{l-\frac{l^2}{2\,x}} \cdot x^{l-x-\frac{1}{2}} \cdot \left(1 + \frac{l}{2\,x} - \frac{l^3}{6\,x^2}\right)\;; \tag{14}$$

equally we receive[7]:

$$(x'+l)^{-l-x'-\frac{1}{2}} \;=\; c^{-l-\frac{l^2}{2\,x'}} \cdot x'^{-l-x'-\frac{1}{2}} \cdot \left(1 - \frac{l}{2x'} + \frac{l^3}{6\,x'^2}\right). \tag{15}$$

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[4]Laplace approaches $\mathrm{e}^x \approx 1 + x$, more consequent is due to [1910Mel], equation (120), page 335:
$\ln\,\Gamma(n+1) \;=\; -C\,n \;-\; \frac{1}{2\,\pi\,\mathrm{i}} \int_{\frac{3}{2}-\mathrm{i}\,\infty}^{\frac{3}{2}+\mathrm{i}\,\infty} \frac{\pi}{\sin(\pi\,z)}\,\zeta(z)\,\frac{n^z}{z}\,\mathrm{d}z \;=\; -C\,n \;+\; \sum_{\mu=1}^{-\infty} \mathrm{res}_{z\to\mu}\left(\Gamma(-z)\,\Gamma(z)\,\zeta(z)\,n^z\right) \;=$
$-\,C\,n \;+\; (n\,\ln(n) - n + C\,n) \;+\; \left(\frac{1}{2}\,\ln(2\,\pi\,n)\right) \;+\; \left(\frac{1}{12\,n}\right) \;+\; \ldots \;=\; \left(n + \frac{1}{2}\right)\,\ln(n) - n + \frac{1}{2}\,\ln(2\,\pi) + \frac{1}{12\,n} + \ldots,$
thus following: $1 \cdot 2 \cdot 3 \cdots n \;=\; n! \;=\; \Gamma(n+1) \;=\; \sqrt{2\,\pi\,n}\,\left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}^{\frac{1}{12\,n}} \cdots \;=\; n^{n+\frac{1}{2}} \cdot \mathrm{e}^{-n} \cdot \sqrt{2\,\pi} \cdot \mathrm{e}^{\left(\frac{1}{12\,n}+\ldots\right)}$.

[5]today's corrections:
$\frac{1}{1 \cdot 2 \cdot 3 \cdots (x-l)} \;=\; (x-l)^{l-x-\frac{1}{2}} \cdot \frac{\mathrm{e}^{x-l}}{\sqrt{2\,\pi}} \cdot \mathrm{e}^{\left(-\frac{1}{12 \cdot (x-l)} - \cdots\right)}$
$\frac{1}{1 \cdot 2 \cdot 2 \cdots (x'+l)} \;=\; (x'+l)^{-x'-l-\frac{1}{2}} \cdot \frac{\mathrm{e}^{x'+l}}{\sqrt{2\,\pi}} \cdot \mathrm{e}^{\left(-\frac{1}{12 \cdot (x'+l)} - \cdots\right)}$

[6]today's correction: $(x-l)^{l-x-\frac{1}{2}} \;=\; \mathrm{e}^{l-\frac{l^2}{2\,x}} \cdot x^{l-x-\frac{1}{2}} \cdot \mathrm{e}^{\left(\frac{l}{2\,x} - \frac{l^3}{6\,x^2}\right)}$

[7]today's correction: $(x'+l)^{-l-x'-\frac{1}{2}} \;=\; \mathrm{e}^{-l-\frac{l^2}{2\,x'}} \cdot (x')^{-l-x'-\frac{1}{2}} \cdot \mathrm{e}^{\left(-\frac{l}{2x'} + \frac{l^3}{6\,(x')^2}\right)}$

By the preceding we get $p = \frac{x+s}{n+1}$, with the result, that s is less than the unity; therefore by setting $p = \frac{x-z}{n}$, z is between the ranges $\frac{x}{n+1}$ and $-\frac{n-x}{n+1}$, and as a result, is less than unity, apart from the sign. The value of p results $1 - p = \frac{x'+z}{n}$;

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therefore, by the preceding examination, we receive[8]:

$$p^{x-l} \cdot (1-p)^{x'+l} = \frac{x^{x-l} \cdot x'^{x'+l}}{n^n} \cdot \left(1 + \frac{n\,z \cdot l}{x\,x'}\right) ; \tag{16}$$

of which follows[9]:

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots (x-l) \cdot 1 \cdot 2 \cdot 3 \cdots (x'+l)} \cdot p^{x-l} \cdot (1-p)^{x'+l}$$
$$= \frac{\sqrt{n} \cdot c^{-\frac{n\,l^2}{2\,x\,x'}}}{\sqrt{\pi} \cdot \sqrt{2\,x\,x'}} \cdot \left(1 + \frac{n\,z\,l}{x\,x'} + \frac{l\,(x'-x)}{2\,x\,x'} - \frac{l^3}{6\,x^2} + \frac{l^3}{6\,x'^2}\right) . \tag{17}$$

We receive the term preceding the greatest one, being away from this by the distance l, then we set l to be negative in this equation; afterwards we add both terms. Their sum is[10]:

$$\frac{2 \cdot \sqrt{n}}{\sqrt{\pi} \cdot \sqrt{2\,x\,x'}} \cdot c^{-\frac{n\,l^2}{2\,x\,x'}} . \tag{18}$$

If we choose the case $l = 0$ contained therein, the concluding integral is[11]:

$$\sum \frac{2 \cdot \sqrt{n}}{\sqrt{\pi} \cdot \sqrt{2\,x\,x'}} \cdot c^{-\frac{n\,l^2}{2\,x\,x'}} . \tag{19}$$

Therefore this integral expresses the sum of all terms of the binomial[12] $[p + (1-p)]^n$ and is between both terms, of which one[13] has got p^{x+l} as its factor, while the other one owns the factor p^{x-l}, and which therefore both are at the same distance to the greatest term[14]; however, from this sum we must substract the greatest term, which consequently is contained twice[15].

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[8] today's syntax for $z^2 \approx 0$ results: $p^{x-l} \cdot (1-p)^{x'+l} \approx \frac{x^{x-l} \cdot (x')^{x'+l}}{n^n} \cdot \left(1 + \frac{n \cdot z \cdot l}{x \cdot x'}\right)$

[9] today's correction:
$\frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots (x-l) \cdot 1 \cdot 2 \cdot 3 \cdots (x'+l)} \cdot p^{x-l} \cdot (1-p)^{x'+l} \approx \frac{\sqrt{n} \cdot \mathrm{e}^{-\frac{n \cdot l^2}{2 \cdot x \cdot x'}}}{\sqrt{2 \cdot \pi \cdot x \cdot x'}} \cdot \left(1 + \frac{n \cdot z \cdot l}{x \cdot x'}\right) \cdot \mathrm{e}^{\left(\frac{l \cdot (x'-x)}{2 \cdot x \cdot x'} - \frac{l^3}{6 \cdot x^2} + \frac{l^3}{6 \cdot x'^2}\right)}$

[10] today's correction with $\lim_{n\to\infty} \frac{n}{x \cdot x'} = \lim_{n\to\infty} \frac{n}{n \cdot p \cdot n \cdot (1-p)} = \lim_{n\to\infty} \frac{1}{n \cdot p \cdot (1-p)} = 0$ results:
$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{\pi} \cdot \sqrt{2 \cdot x \cdot x'}} \cdot \mathrm{e}^{-\frac{n \cdot l^2}{2 \cdot x \cdot x'}} \cdot \left[\left(1 - \frac{n \cdot z \cdot l}{x \cdot x'}\right) \mathrm{e}^{-\frac{l \cdot n \cdot (2 \cdot p - 1)}{x \cdot x'} + 0} + \left(1 + \frac{n \cdot z \cdot l}{x \cdot x'}\right) \mathrm{e}^{\frac{l \cdot n \cdot (2 \cdot p - 1)}{x \cdot x'} + 0}\right] = 0$

[11] today's correction: $\lim_{n\to\infty} \left[\sum_{l=-x'}^{x} \frac{\sqrt{n} \cdot \mathrm{e}^{-\frac{n \cdot l^2}{2 \cdot x \cdot x'}}}{\sqrt{\pi \cdot 2 \cdot x \cdot x'}} + \sum_{l=-x}^{n-x=x'} \frac{\sqrt{n} \cdot \mathrm{e}^{-\frac{n \cdot l^2}{2 \cdot x \cdot x'}}}{\sqrt{\pi \cdot 2 \cdot x \cdot x'}}\right] = 2 \cdot \infty \cdot 0 = 2 \cdot 1 = 2$

[12] For all n is valid: $[p + (1-p)]^n = 1^n = 1$.

[13] Correct: $(1-p)^{x'+l}$, symmetry is valid for $p = \frac{1}{2}$ with $x = n \cdot p = \frac{n}{2} = n \cdot (1-p) = x'$ only.
Here Laplace comes erraneously from a sum, which in the limit $n \to \infty$ would run from $l = 0$ until ∞. At this point he does not remark his error, because he just indicates the Leibniz notation only, by omitting the sum limits. By mirroring at the expectation value with subsequent addition, he forces a symmetry, which else mainly exists for the limit zero at $n \to \infty$.

[14] Here, the maximum becomes zero by the limit $n \to \infty$.

[15] Not only the greatest term itself, but even the whole sum is doubled.

To receive this definite integral, due to No. 10 in the first book, we now consider y to be a function of l with[16]:

$$\sum y \,=\, \frac{1}{c^{\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)} \,-\, 1} \,=\, \left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)^{-1} \,-\, \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)^{0} \,+\, \frac{1}{12}\,\frac{\mathrm{d}y}{\mathrm{d}l} \,+\, \text{etc.}\,; \tag{20}$$

from which we can derive by the same number[17]:

$$\sum y \,=\, \int y\,\mathrm{d}l \,-\, \frac{1}{2}\,y \,+\, \frac{1}{12}\,\frac{\mathrm{d}y}{\mathrm{d}l} \,+\, \text{etc.} \,+\, \text{constant}\,. \tag{21}$$

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By equating here y to[18] $\frac{2\cdot\sqrt{n}}{\sqrt{\pi}\cdot\sqrt{2\,x\,x'}}\cdot c^{-\frac{n\,l^2}{2\,x\,x'}}$, the subsequent differentials of y take the factor $\frac{n\,l}{2\,x\,x'}$ and its abilities; thus preceded, that l cannot be greater than the order $\sqrt{n}$, this factor is of the order $\frac{1}{\sqrt{n}}$, and consequently its differentials decrease more and more, devided by the corresponding powers of $\mathrm{d}l$; therefore, by neglecting the terms of the order $\frac{1}{n}$, analogous to the preceding, we receive[19]:

$$\sum y \,=\, \int y\,\mathrm{d}l \,-\, \frac{1}{2}\,y \,+\, \frac{1}{2}\,Y\,, \tag{22}$$

by beginning with l both definite and infinitely small integrals and by calling Y the greatest term of the binomial[20].

The sum of all terms of the binomial[21] $[p \,+\, (1\,-\,p)]^n$, which are contained between both terms, and which are both equally distant to the greatest term of number l, equated to[22] $\sum y \,-\, \frac{1}{2}\,Y$, results[23]:

$$\int y\,\mathrm{d}l \,-\, \frac{1}{2}\,y\,; \tag{23}$$

and if the sum of both of these most outside terms[24] is added here, then we receive as sum of all of these terms[25]:

$$\int y\,\mathrm{d}l \,+\, \frac{1}{2}\,y\,. \tag{24}$$

By setting:

$$t \,=\, \frac{l\,\sqrt{n}}{\sqrt{2\,x\,x'}}\,, \tag{25}$$

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[16] today's correction: $\sum_{l\,=\,-x}^{n\,-\,x\,=\,x'}\; y(l) \,\neq\, \frac{1}{\mathrm{e}^{\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)}\,-\,1} \,=\, \left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)^{-1} \,-\, \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)^{0} \,+\, \frac{1}{12}\,\frac{\mathrm{d}y}{\mathrm{d}l} \,+\, \text{etc.}$

[17] Here, Laplace shows that he has not understood and applied consequently enough the Leibniz notation of differential calculation. Today's correction: $\sum_{l\,=\,-x}^{n\,-\,x\,=\,x'}\; y(l) \,\to\, \int_{-\infty}^{\infty} y(l)\,\mathrm{d}l \,\neq\, \frac{\mathrm{d}l}{\mathrm{d}y} \,-\, \frac{1}{2} \,+\, \frac{1}{12}\,\frac{\mathrm{d}y}{\mathrm{d}l} \,+\, \text{etc.}$

[18] today's correction: $\frac{\sqrt{n}}{\sqrt{2\,\pi\,x\,x'}}\cdot\mathrm{e}^{-\frac{n\,l^2}{2\,x\,x'}}$

[19] today's correction: $\sum_{l\,=\,-x}^{n\,-\,x\,=\,x'}\; y(l) \,\to\, \int_{-\infty}^{\infty} y(l)\,\mathrm{d}l \,=\, 1$

[20] today's correction: determining the definite and infinitely small integral

[21] For all n is valid: $1^n \,=\, \mathrm{e}^{n\,\ln(1)} \,=\, \mathrm{e}^0 \,=\, 1$.

[22] today's correction: $\sum_{l\,=\,-x}^{n\,-\,x\,=\,x'}\; y(l)$

[23] today's correction: $\int_{-\infty}^{\infty} y(l)\,\mathrm{d}l$

[24] They are as good as zero.

[25] today's correction: $\int_{-\infty}^{\infty} y(l)\,\mathrm{d}l$

we receive the sum[26]:

$$\frac{2}{\sqrt{\pi}} \cdot \int \mathrm{d}t \cdot c^{-t^2} + \frac{\sqrt{n}}{\sqrt{\pi} \cdot \sqrt{2\,x\,x'}} \cdot c^{-t^2} \cdot (0)\,. \tag{26}$$

Presupposed that the terms we neglected of this[27] belong to the order $\frac{1}{n}$, the preceding expression becomes the more precise[28], the more n is increased: It is valid strictly, if n is infinite[29]. By the preceding analysis it should be easy to consider the terms of the order $\frac{1}{n}$ and higher orders.

References

[1814Lapl] (Pierre Simon) de Laplace: *Théorie analytique des probabilités (Analytic Theory of Probabilities)*, Courcier, Paris, 2nd edition, (1814), copy by Bayerische StaatsBibliothek, Münchner Digitalisierungszentrum: http://reader.digitale-sammlungen.de/de/fs1/object/display/bsb10053604_00007.html on 06/16/2017

[1910Mel] (Hjalmar) Mellin: Abriß einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen (Outline on an Unified Theory of the Gamma and the Hypergeometric Functions), *Mathematische Annalen*, **68**, (1910), 305–337

[26] today's correction: $\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\, \frac{\mathrm{d}t}{\sqrt{\pi}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \mathrm{e}^{-u}\, \frac{\mathrm{d}u}{2\sqrt{u}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} = \sqrt{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)}{\pi}} = \frac{1}{\sqrt{\sin\left(\frac{\pi}{2}\right)}} = 1$

[27] This means, from the preceding sum.

[28] today's correction: The norm 1 is fulfilled strictly, independently of n .

[29] Here, for $n \to \infty$ is valid: $\infty \cdot 0 = 1$.